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BROWNIAN FLUCTUATIONS IN GALVANOMETERS AND GALVANOMETER AMPLIFIERS

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CONTENTS

The Brownian fluctuations in moving-coil galvanometers and galvanometer amplifiers have been investigated theoretically and experimentally. Previous work is summarized and a simple treatment given of the fluctuations of a galvanometer with negligible inductance in its circuit; here, as elsewhere in the paper, the correlation function of the random force is used, not the frequency spectrum. Both molecular bombardment of the suspended mirror and Johnson noise in the circuit resistance are considered, and a comment is made on why the coexistence of these two effects does not increase the r.m.s. deflexion above its equipartition value. It is further shown by an exact random force calculation that the presence of an appreciable inductance in the galvanometer circuit does not change the r.m.s. values of deflexion and angular velocity. The same result is obtained by a statistical mechanical argument based on the assumptions already implicit in the application of the equipartition principle to, say, a suspended mirror.

Expressions are obtained for the magnitude and correlation function of the Brownian fluctuations of a galvanometer amplifier with two galvanometers of arbitrary periods and damping; these expressions are in terms of the periods and damping constants of the two galvanometers, the mechanical damping of the primary galvanometer and the resistance of the primary circuit. A method of finding the magnitude and correlation function of the fluctuations from a record of the throw of the secondary galvanometer consequent on passing a known charge through the primary is suggested.

The magnitude of the Brownian fluctuations in a galvanometer amplifier was determined experimentally. The effects of external disturbances were reduced till the changes in zero, including overall drift, caused by them in 30 min. (say 1000 times the response time) were much less than the r.m.s. Brownian deflexion. The methods used to determine the quantities in the theoretical expression for the r.m.s. Brownian deflexion are described. Various conditions of damping were used and the average ratio of experimental to theoretical r.m.s. deflexions was found to be 100·0 % with ± 1·1 % standard error.

An appendix describes the verification for the case of the galvanometer amplifier of Rice's formulae for the number of zeros and number of points of zero slope on the record of a one-dimensional Gaussian random process.

Introduction

The investigations described in this paper are intended to establish, both theoretically and experimentally, the limit of observation set upon moving-coil galvanometers and galvanometer amplifiers by thermal agitation. One of us (C. W. McC.) has been responsible for the theory, the other for the experiments.

Summary of earlier investigations

Ising in 1926 discussed the Brownian motion of a galvanometer by treating it as a system of one degree of freedom and applying the equipartition theorem. He assumed that the root-mean-square deflexion must correspond to a potential energy of $\frac{1}{2}kT$ in the suspension. From this he showed that the disturbances in a galvanometer zero trace published by Moll & Burger in 1925 were of the order to be expected from thermal fluctuations. Zernike, also in 1926, obtained the same result as Ising by considering current fluctuations only.

In 1927 Ornstein and his collaborators treated the Brownian motion of a galvanometer as arising from a random e.m.f. in the galvanometer circuit together with an independent random couple due to the bombardment of the suspended system by air molecules. Making assumptions about these random forces which amounted to applying the equipartition principle to simple systems, Ornstein and his collaborators obtained Ising's result for the critically damped galvanometer. They also investigated experimentally the magnitudes of the fluctuations of such a galvanometer when connected to a circuit at room temperature and then at the temperature of liquid air. The observed ratio of the r.m.s. magnitudes at the two temperatures compared well with that forecast by the theory, but no absolute measurement was made. This theory was extended in 1933 by van Lear, using the concept of the spectrum of a random force; he showed that Ising's result still held even when the galvanometer was not critically damped. Van Lear also treated the case of the galvanometer amplifier when both galvanometers have the same period.

In 1932, Ising described experiments with a galvanometer, where he recorded the fluctuations as observed directly with a microscope. The magnification had to be so high that diffraction bands masked the image of the galvanometer needle; the records showed a drifting diffraction pattern on which was superimposed the Brownian fluctuation. Ising estimated the r.m.s. of the fluctuations to be within 5 % of his theoretical value. In the same year Barnes & Matossi published observations with a galvanometer amplifier showing that it was near the Ising limit, but the number of observations was unduly small.

Strong (1948) and Astbury (1948) have since questioned Ising's result. Astbury argues that there are two independent causes of fluctuation, the Nyquist e.m.f. and molecular bombardment; each of these must by itself be assumed to give Ising's result; the overall r.m.s. fluctuation obtained by adding errors in the normal manner should therefore be $\sqrt{2}$ times Ising's result. Astbury also states that experimentally he has not been nearer the Ising limit than a factor of two. As against these arguments, Surdin (1949) has given a full theoretical derivation of Ising's result for a single galvanometer using the concept of the spectrum of a random force.

In 1938 Moullin suggested that an inductance in a galvanometer circuit might alter the r.m.s. Brownian deflexion. On the Ising approach this suggestion implies that electromagnetic energy may have to be added to the torsional energy before equating to $\frac{1}{2}kT$; this statistical-mechanical point of view is considered briefly later. In 1936 Niessen, applying the random-force method to a critically damped galvanometer with inductance in its circuit, gave a result differing from Ising's by terms of order $L\omega/R$, but his argument involved an approximation. It will be shown later that, on the assumptions of the random force method, Ising's value for the mean potential energy of the suspension holds independently of any inductance in the circuit, and that the corresponding result for the mean kinetic energy holds also.

Objects of the present paper

The present theoretical investigations cover the Brownian motion of a galvanometer with an appreciable inductance in its circuit, and of a galvanometer amplifier containing two galvanometers, not necessarily of the same period, with arbitrary conditions of damping.

The Brownian motion of the galvanometer amplifier will be considered using the random force method; no simple statistical-mechanical argument is available in this case. The general case will be treated and the correlation function for the motion of the secondary galvanometer obtained. Since this work was completed a discussion by Passoth (1942) of the Brownian motion of the galvanometer amplifier has come to our notice. By Fourier analysis he obtained an expression for the mean-square deflexion when the secondary galvanometer is critically damped. The general result given here reduces to his expression in this special case.

It will be shown that the amplifier can be calibrated by passing a known charge rapidly through the primary galvanometer and recording the ensuing throw in the secondary galvanometer. To predict the magnitude and autocorrelation function of the Brownian fluctuation it is then necessary to measure in addition only the primary circuit resistance, and, as a small correcting term, the ratio of mechanical to total damping in the primary galvanometer.

Throughout the paper the arguments are based on the elementary properties of the correlation functions of the random forces considered; the idea of the spectrum of a random force, which has been used in other recent investigations, has not been introduced. The two methods are essentially equivalent, but for linear systems the correlation function arguments seem more elementary and natural, especially when a correlation function is the aim of the calculation.

Since Ornstein's classic investigation determined only the ratio of the fluctuations at two temperatures and not their absolute magnitude, and recent advances in technique have made it possible to improve on the records of Ising and of Moll & Burger, we have undertaken new experiments to determine the magnitude of the thermal fluctuations of a galvanometer amplifier. These experiments have covered various conditions of damping, and have confirmed Ising's result and our own theoretical investigations. The experiments have also shown that the theoretical limit can be consistently attained in practice.

Part I. Theory

Outline of the 'random-force' method

It will be convenient to derive briefly the well-known properties of random forces which are required later.

Consider first the Brownian fluctuations in current i in a circuit having inductance L and resistance R. It is assumed that the current is due to a random e.m.f. $E(t)$ appearing in the circuit. Then the circuital equation is

$$L\,di/dt+Ri=E(t), \tag{1}$$

the time average $\overline{E(t)}$ being zero. Moreover, since $E(t)$ must be regarded as originating in a very large number of virtually independent events on an atomic scale, the values of E at two instants separated by an interval τ may be assumed to be uncorrelated if τ is large compared with the time-scale of atomic processes. Accordingly, it will be assumed that $\overline{E(t)\,E(t+\tau)}$ is zero unless τ is negligibly small compared with, for example, the time constant of the circuit.

The next step is to determine what further assumptions must be made about $E(t)$ in order to ensure that $\frac{1}{2}L\overline{i^2}$ has its equipartition value $\frac{1}{2}kT$. The solution of (1) which vanishes when t is zero is

$$i=(1/L)\,\mathrm{e}^{-(R/L)t}\int_0^t \mathrm{e}^{(R/L)x}\,E(x)\,dx. \tag{2}$$

The boundary condition used may be decided by convenience, as one is interested in the solution only when the arbitrary initial condition has ceased to matter, i.e. as t tends to infinity. From (2)

$$i^2=(1/L^2)\,\mathrm{e}^{-2(R/L)t}\int_0^t\int_0^t \mathrm{e}^{(R/L)(x+y)}\,E(x)\,E(y)\,dx\,dy. \tag{3}$$

Putting $y=x+\tau$ and changing variables to x and τ, one gets

$$i^2=(1/L^2)\,\mathrm{e}^{-2(R/L)t}\int_0^t\int_{-x}^{t-x} \mathrm{e}^{(R/L)(2x+\tau)}\,E(x)\,E(x+\tau)\,d\tau\,dx. \tag{4}$$

The average is now taken over a large number of identical circuits in all of which the current is initially zero. According to the assumptions made above, if $\overline{E(x)\,E(x+\tau)}$ appears in the integrand of an integral over a range of τ including $\tau=0$ as an internal point, we may, without altering the value of the integral appreciably, extend the range of integration to infinity in both directions, and replace any other relatively slowly varying factors in the integrand by their values at $\tau=0$. It should be noticed that, here and subsequently, treating $\overline{E(t)\,E(t+\tau)}$ as essentially a δ-function is regarded as an approximation, albeit a very good one. If it were asserted that $\overline{E(t)\,E(t+\tau)}$ was a δ-function, it would be inconsistent to regard $E(t)$ as differentiable as will be done later (cf. Moyal 1949). Then, if it is allowed that the approximation is so good as to justify the use of a sign of equality, (4) gives

$$\overline{i^2}=(1/L^2)\,\mathrm{e}^{-2(R/L)t}\int_0^t \mathrm{e}^{2(R/L)x}\int_{-\infty}^{\infty}\overline{E(x)\,E(x+\tau)}\,d\tau\,dx. \tag{5}$$

Since $\overline{E(x)\,E(x+\tau)}$ is clearly independent of x, its integral over all τ is a constant, which will be denoted by $\overline{EE}$. Then

$$\lim_{t\to\infty}\overline{i^2}=\overline{EE}/2RL. \tag{6}$$

The average here is still an ensemble average, but is one taken when the arbitrary initial conditions imposed on all members of the ensemble have ceased to matter. This average will be the same as a time average taken on a single circuit for which initial conditions have ceased to matter. The present investigation deals particularly with averages of this latter sort. Considering the result (6), one easily sees that the necessary and sufficient further condition that the equipartition theorem should be satisfied is

$$\overline{EE} = 2RkT. \tag{7}$$

It will be assumed that there is a random e.m.f. in the circuit of the galvanometer with all these properties. R is the total resistance in the galvanometer circuit.

Consider next a suspended system without torsional control, having moment of inertia I and damping constant κ. The fluctuations are assumed due to a random couple $F(t)$, where $\overline{F(t)} = 0$ and $\overline{F(t)\,F(t+\tau)} = 0$ unless τ is small, as explained in the discussion of $E(t)$. If ω denotes the angular velocity of the system, one has

$$I\,d\omega/dt + \kappa\omega = F(t). \tag{8}$$

Proceeding as previously, assuming that $\frac{1}{2}I\overline{\omega^2}$ must equal $\frac{1}{2}kT$, and denoting the integral of $\overline{F(t)\,F(t+\tau)}$ over all τ by $\overline{FF}$, one obtains

$$\overline{FF} = 2\kappa kT. \tag{9}$$

It can be shown that the random couple has the property (9) even when there is torsional control. It is assumed that a random couple with these properties acts on the suspended system of the galvanometer, this couple being independent of that arising indirectly from the random e.m.f. in the coil, so that even if $t_1 = t_2$

$$\overline{F(t_1)\,E(t_2)} = 0. \tag{10}$$

The properties of the random forces in both the cases considered above have been obtained also by using detailed models (see Uhlenbeck & Goudsmit 1929; and Lawson & Uhlenbeck 1950).

Galvanometer with negligible inductance

The simple case of a galvanometer with negligible inductance will now be discussed to exemplify the method to be applied to more complicated cases. It leads to a comment on the physical basis of the apparent non-additive property of the two random forces considered; this comment is akin to a remark of Zernike (1926).

The galvanometer is assumed to be of the suspended-coil type. The flux linkage of the coil is denoted by G; c is the torsional constant. Then, assuming that the galvanometer deflexion, θ, is proportional to the current, the equations of the Brownian motion are

$$I\,d^2\theta/dt^2 + \kappa\,d\theta/dt + c\theta = F(t) + Gi, \tag{11a}$$

$$L\,di/dt + Ri = E(t) - G\,d\theta/dt, \tag{11b}$$

$$\overline{FF} = 2\kappa kT, \quad \overline{EE} = 2RkT, \quad \overline{E(t_1)\,F(t_2)} = 0. \tag{11c}$$

Neglecting L, one finds

$$I\,d^2\theta/dt^2 + (\kappa + G^2/R)\,d\theta/dt + c\theta = F(t) + (G/R)\,E(t) \tag{12}$$

or

$$(D+\alpha_1)\,(D+\alpha_2)\,\theta = (1/I)\,\{F(t) + (G/R)\,E(t)\}, \tag{13}$$

where D denotes d/dt and

$$\alpha_1 + \alpha_2 = (1/I)\,(\kappa + G^2/R), \quad \alpha_1\alpha_2 = c/I, \tag{14}$$

the real parts of α_1 and α_2 being positive. The convenient solution of (13) is

$$\theta = \frac{e^{-\alpha_1 t}}{I(\alpha_2-\alpha_1)}\int_0^t e^{\alpha_1 x}\{F(x) + (G/R)\,E(x)\}\,dx + \frac{e^{-\alpha_2 t}}{I(\alpha_1-\alpha_2)}\int_0^t e^{\alpha_2 x}\{F(x) + (G/R)\,E(x)\}\,dx. \tag{15}$$

Proceeding as in the derivation of (6) from (2), and using the symbol $\mathfrak{W}$ to denote the mean value of the expression which follows it, one can show that

$$\lim_{t\to\infty} \mathfrak{W}\left[\mathrm{e}^{-\alpha_r t}\int_0^t \mathrm{e}^{\alpha_r x} P(x)\,dx\,.\,\mathrm{e}^{-\alpha_s t}\int_0^t \mathrm{e}^{\alpha_s x} P(x)\,dx\right] = \frac{\overline{PP}}{\alpha_r+\alpha_s}, \tag{16}$$

where P is either F or E. Moreover, the result

$$\lim_{t\to\infty} \mathfrak{W}\left[\mathrm{e}^{-\alpha_r t}\int_0^t \mathrm{e}^{\alpha_r x} F(x)dx\,.\,\mathrm{e}^{-\alpha_s t}\int_0^t \mathrm{e}^{\alpha_s x} E(x)\,dx\right] = 0 \tag{17}$$

is established by writing the left-hand side as a double integral and at once applying the last relation in (11 *c*).

Using (16) and (17) one then obtains from (15)

$$\lim_{t\to\infty}\overline{\theta^2} = \frac{\overline{FF}+(G^2/R^2)\,\overline{EE}}{I^2(\alpha_2-\alpha_1)^2}\left\{\frac{1}{2\alpha_1}+\frac{1}{2\alpha_2}-\frac{2}{\alpha_1+\alpha_2}\right\}$$
$$= \frac{\overline{FF}+(G^2/R^2)\,\overline{EE}}{2I^2\alpha_1\alpha_2(\alpha_1+\alpha_2)}. \tag{18}$$

Then the relations (14) give

$$\lim_{t\to\infty}\overline{\theta^2} = \frac{\overline{FF}+(G^2/R^2)\,\overline{EE}}{2c(\kappa+G^2/R)}. \tag{19}$$

Substituting for $\overline{FF}$ and $\overline{EE}$ from (11 *c*) then gives Ising's result, $\frac{1}{2}c\overline{\theta^2}$ equals $\frac{1}{2}kT$.

Invariance of r.m.s. deflexion on adding sources of fluctuation

The form of (19) is instructive. From (19) and (11 *c*) one can obtain Ising's result by considering, for example, only the random couple, provided that only the mechanical damping is considered, the electromagnetic damping and the random e.m.f. being ignored. This arises from the fact that $\overline{FF}$ and $(G^2/R^2)\,\overline{EE}$ are proportional to κ and G^2/R respectively, which is substantially Zernike's explanation of why his calculations give the same results as Ising's. It is worth stressing this point, as it gets at once to the origin of the confusion which exists on this subject. If one desires a picture of what is happening, $\overline{EE}$ and $\overline{FF}$ may be regarded as measures of the tendency of the corresponding random forces to produce fluctuations in current and deflexion respectively. The form of the numerator of (19) then indicates, as might be intuitively expected, that the tendencies of the two random forces to impart mean energy to the coil are indeed additive; but the fact that the total damping term, which is also additively constructed, appears in the denominator implies that the resultant mean potential energy stored in the suspension is determined by the ratio of the total tendency to deflect to the total damping. The tendency of each source of fluctuation to produce a deflexion is automatically related to the damping that it introduces; the relation ensures that the mean energy stored in the suspension cannot be changed by adding new sources of fluctuation, providing that each source operates at the same temperature, and that the equipartition principle holds. As a concrete example, the admission of air to a previously evacuated galvanometer does not change its r.m.s. fluctuation, since the tendency of the molecular bombardment to increase the r.m.s. deflexion is exactly counterbalanced by the viscous air-damping automatically introduced.

Galvanometer with inductance

If (11 *a*) is differentiated once with respect to time and then i and di/dt eliminated between the resulting equation and (11 *a*) and (11 *b*), one obtains

$$\{LID^3+(\kappa L+RI)\,D^2+(cL+G^2+R\kappa)\,D+Rc\}\,\theta = GE(t)+RF(t)+L\dot{F}(t). \tag{20}$$

This may be written

$$(D+\alpha_1)\,(D+\alpha_2)\,(D+\alpha_3)\,\theta = (1/LI)\,\{GE(t)+RF(t)+L\dot{F}(t)\}, \tag{21}$$

where

$$\left.\begin{aligned}\Sigma\alpha_1 &= (1/LI)\,(\kappa L+RI),\\ \Sigma\alpha_1\alpha_2 &= (1/LI)\,(cL+G^2+R\kappa),\\ \alpha_1\alpha_2\alpha_3 &= Rc/LI.\end{aligned}\right\} \tag{22}$$

Since the coefficients are all positive and

$$(\kappa L+RI)\,(cL+G^2+R\kappa) > RcLI,$$

α_1, α_2 and α_3 all have their real parts positive; otherwise, (20) would imply instability.

Write

$$\left.\begin{aligned}\chi(z) &= (z-\alpha_1)\,(z-\alpha_2)\,(z-\alpha_3),\\ \chi'(z) &= d\chi/dz.\end{aligned}\right\} \tag{23}$$

A convenient solution of (20) is

$$\theta = (1/LI)\sum_{r=1}^{3}\{1/\chi'(\alpha_r)\}\,e^{-\alpha_r t}\int_0^t e^{\alpha_r x}\{GE(x)+RF(x)+L\dot{F}(x)\}\,dx. \tag{24}$$

Integrating by parts to eliminate $\dot{F}$, one obtains

$$e^{-\alpha_r t}\int_0^t e^{\alpha_r x}\{GE(x)+RF(x)+L\dot{F}(x)\}\,dx$$
$$= e^{-\alpha_r t}\int_0^t e^{\alpha_r x}\{GE(x)+(R-L\alpha_r)\,F(x)\}\,dx+LF(t)-L\,e^{-\alpha_r t}F(0). \tag{25}$$

Since t is to tend to infinity the last term in (25) may be ignored. Also

$$\sum_{r=1}^{3}\{1/\chi'(\alpha_r)\} = 0, \tag{26}$$

so (24) becomes

$$\theta = (1/LI)\sum_{r=1}^{3}\{1/\chi'(\alpha_r)\}\,e^{-\alpha_r t}\int_0^t e^{\alpha_r x}\{GE(x)+(R-L\alpha_r)\,F(x)\}\,dx. \tag{27}$$

From (16), (17)

$$\lim_{t\to\infty}\overline{\theta^2} = \frac{1}{L^2I^2}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{G^2\overline{EE}+R^2\overline{FF}+L^2\alpha_r\alpha_s\overline{FF}}{\chi'(\alpha_r)\,\chi'(\alpha_s)\,(\alpha_r+\alpha_s)}+\frac{R\overline{FF}}{LI}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{1}{\chi'(\alpha_r)\,\chi'(\alpha_s)}. \tag{28}$$

By (26) the last member of (28) is zero. Using the identities

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{1}{\chi'(\alpha_r)\,\chi'(\alpha_s)\,(\alpha_r+\alpha_s)} = \frac{1}{2}\frac{\Sigma\alpha_1}{\alpha_1\alpha_2\alpha_3\{\Sigma\alpha_1\,\Sigma\alpha_1\alpha_2-\alpha_1\alpha_2\alpha_3\}}, \tag{29}$$

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\alpha_r\alpha_s}{\chi'(\alpha_r)\,\chi'(\alpha_s)\,(\alpha_r+\alpha_s)} = \frac{1}{2}\frac{1}{\Sigma\alpha_1\,\Sigma\alpha_1\alpha_2-\alpha_1\alpha_2\alpha_3}, \tag{30}$$

(28) may be written

$$\lim_{t\to\infty}\overline{\theta^2} = \frac{1}{2L^2I^2}\frac{(G^2\overline{EE}+R^2\overline{FF})\,\Sigma\alpha_1+L^2\overline{FF}\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3\{\Sigma\alpha_1\,\Sigma\alpha_1\alpha_2-\alpha_1\alpha_2\alpha_3\}}. \tag{31}$$

Substituting for $\overline{EE}$ and $\overline{FF}$ and using (22), (31) becomes

$$\lim_{t\to\infty} \overline{\theta^2} = kT/c, \tag{32}$$

which reproduces Ising's result. To obtain $\overline{\dot{\theta}^2}$, one differentiates (27)

$$\dot{\theta} = -\frac{1}{LI}\sum_{r=1}^{3}\frac{\alpha_r}{\chi'(\alpha_r)}\mathrm{e}^{-\alpha_r t}\int_0^t \mathrm{e}^{\alpha_r x}\{GE(x)+(R-L\alpha_r)F(x)\}\,dx+\sum_{r=1}^{3}\frac{GE(t)+(R-L\alpha_r)F(t)}{LI\chi'(\alpha_r)}. \tag{33}$$

The second sum vanishes, by (26), and

$$\sum_{r=1}^{3}\{\alpha_r/\chi'(\alpha_r)\} = 0. \tag{34}$$

Proceeding as before, one gets

$$\lim_{t\to\infty}\overline{\dot{\theta}^2} = \frac{1}{L^2I^2}\sum_{r=1}^{3}\frac{(G^2\overline{EE}+R^2\overline{FF})\,\alpha_r\alpha_s+LR\alpha_r\alpha_s(\alpha_r+\alpha_s)\,\overline{FF}+L^2\alpha_r^2\alpha_s^2\,\overline{FF}}{\chi'(\alpha_r)\,\chi'(\alpha_s)\,(\alpha_r+\alpha_s)}. \tag{35}$$

In this case the algebraic identities required are (30), (34) and

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\alpha_r^2\alpha_s^2}{\chi'(\alpha_r)\,\chi'(\alpha_s)\,(\alpha_r+\alpha_s)} = \frac{1}{2}\frac{\Sigma\alpha_1\alpha_2}{\Sigma\alpha_1\,\Sigma\alpha_1\alpha_2-\alpha_1\alpha_2\alpha_3}. \tag{36}$$

Proceeding as before, one obtains from (35)

$$\lim_{t\to\infty}\overline{\dot{\theta}^2} = kT/I. \tag{37}$$

Solving (11 *a*) and (11 *b*) for i instead of θ, one can show similarly that $\frac{1}{2}L\overline{i^2}$ has the value $\frac{1}{2}kT$.

Application of statistical mechanics

The discussion given in this paper is based on applications of statistical mechanics, in the form of the equipartition theorem, to two simple macroscopic systems, the suspended mirror and the electric circuit with inductance and resistance. The assumptions already implicitly involved in these applications of statistical mechanics are sufficient to determine, independently of the random-force method, the mean-square deflexion of a galvanometer in an inductive circuit.

A consideration of the case of the suspended mirror makes these assumptions clear. Any suspended mirror has, viewed microscopically, a large number of degrees of freedom. It is assumed that one may consider only the subsystem described by the macroscopic variables, θ and $\dot{\theta}$, and, ignoring viscosity, apply the statistical-mechanical principles appropriate to a system having only a small interaction with its surroundings. The equipartition theorem may be applied, since the energy is the sum of square terms.

The same initial assumptions may be applied to the galvanometer and its associated circuit. Here both resistance and viscosity are ignored. The energy of the resulting system expressed in terms of i, θ and $\dot{\theta}$ is not a sum of squares and the equipartition theorem cannot at once be used. Nevertheless, the application of statistical-mechanical principles is straightforward. The kinetic and potential energies E and V are given by

$$E = \tfrac{1}{2}I\dot{\theta}^2+\tfrac{1}{2}Li^2+Gi\theta, \tag{38}$$

$$V = \tfrac{1}{2}c\theta^2. \tag{39}$$

From the resulting Lagrangian one finds that the momentum conjugate to θ, denoted by p_θ, is $I\dot{\theta}$, and the momentum conjugate to the charge passed Q, denoted by p_Q, is $Li+G\theta$.

The Hamiltonian can be written down in terms of these conjugate co-ordinates, but it is simpler in terms of the original co-ordinates:

$$H = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}Li^2 + \tfrac{1}{2}c\theta^2. \tag{40}$$

In this instance, of course, the Hamiltonian does not give the energy of the system, as the energy of interaction with the field does not appear.

One now averages over a canonical ensemble, noting that it is the Hamiltonian and not the energy $(E+V)$ which appears in the exponential term. This method of averaging where there is interaction with a magnetic field has been discussed by Broer (1946). The mean value of any function Φ of the co-ordinates is then given by

$$\overline{\Phi} = \frac{\iiiint \Phi\, \mathrm{e}^{-H/kT}\, dQ\, dp_Q\, d\theta\, dp_\theta}{\iiiint \mathrm{e}^{-H/kT}\, dQ\, dp_Q\, d\theta\, dp_\theta}. \tag{41}$$

Since the original co-ordinates are linear functions of the conjugate ones, the Jacobian of the transformation from the latter to the former is a constant, and will cancel out if this transformation is made in both integrals of (41). One gets, noting that Q does not appear in H, and assuming it does not appear in Φ either,

$$\overline{\Phi} = \frac{\iiint \Phi\, \mathrm{e}^{-H/kT}\, di\, d\theta\, d\dot{\theta}}{\iiint \mathrm{e}^{-H/kT}\, di\, d\theta\, d\dot{\theta}}. \tag{42}$$

From (40) it follows that each of $\tfrac{1}{2}c\overline{\theta^2}$, $\tfrac{1}{2}I\overline{\dot{\theta}^2}$ and $\tfrac{1}{2}L\overline{i^2}$ is equal to $\tfrac{1}{2}kT$, in agreement with the results of other sections.

Brownian fluctuations of the galvanometer amplifier

In the galvanometer amplifier, a deflexion of the primary galvanometer introduces a proportional e.m.f. into the circuit of the secondary galvanometer. This has the advantage that the ensuing deflexion in the secondary can be much larger than that in the primary. In this section information will be obtained about the motion in the secondary arising from Brownian fluctuations in the primary. More specifically, the correlation function, $\overline{\theta(t)\,\theta(t+\tau)}$, for the deflexion, θ, in the secondary galvanometer will be calculated without restrictive assumptions about the relative periods and damping conditions of the two galvanometers. From this an expression for $\overline{\theta^2}$, which is the result chiefly required for the experimental investigation, will be obtained. This expression could be obtained more directly as will be indicated, but the correlation function itself is of interest.

It is assumed that the Brownian motion of the secondary galvanometer itself is negligible, and that in the circuits of both primary and secondary the inductance can be neglected.

Denoting the deflexion of the primary galvanometer by ϕ, the equation of motion takes the form

$$(D^2 + 2\beta_1 D + \omega_1^2)\,\phi = \frac{1}{I_1}F_1(t) + \frac{G_1}{I_1 R_1}E_1(t), \tag{43}$$

where $F_1(t)$ and $E_1(t)$ are the random couple and e.m.f. respectively in the primary galvanometer; $2\beta_1$ is the damping constant, $(2\pi/\omega_1)$ the free period, R_1 the circuit resistance, G_1 the effective flux linkage and I_1 the moment of inertia of that galvanometer. On the assumptions made above, the motion of the secondary will be given by

$$(D^2+2\beta_2 D+\omega_2^2)\,\theta = a\phi, \tag{44}$$

where a is some constant; $2\beta_2$ and $(2\pi/\omega_2)$ are the damping constant and free period of the secondary.

Eliminating ϕ between (43) and (44) one obtains

$$(D^2+2\beta_1 D+\omega_1^2)\,(D^2+2\beta_2 D+\omega_2^2)\,\theta = P(t), \tag{45}$$

where
$$P(t) = \frac{a}{I_1}F_1(t)+\frac{aG_1}{I_1R_1}E_1(t). \tag{46}$$

It follows from the properties of $F_1(t)$ and $E_1(t)$ that

$$\overline{PP} = \int_{-\infty}^{\infty}\overline{P(t)\,P(t+\tau)}\,d\tau = 2kT(a^2/I_1^2)\,(\kappa_1+G_1^2/R_1), \tag{47}$$

where κ_1 is the mechanical damping constant and R_1 the circuit resistance for the primary galvanometer. Obviously

$$(1/I_1)\,(\kappa_1+G_1^2/R_1) = 2\beta_1. \tag{48}$$

Equation (45) can be put in the form

$$(D+\alpha_1)\,(D+\alpha_2)\,(D+\alpha_3)\,(D+\alpha_4)\,\theta = P(t). \tag{49}$$

Putting
$$\omega_1'^2 = \omega_1^2-\beta_1^2,\quad \omega_2'^2 = \omega_2^2-\beta_2^2, \tag{50}$$

we have
$$\left.\begin{aligned}\alpha_1 &= \beta_1+i\omega_1', & \alpha_2 &= \beta_1-i\omega_1',\\ \alpha_3 &= \beta_2+i\omega_2', & \alpha_4 &= \beta_2-i\omega_2'.\end{aligned}\right\} \tag{51}$$

Writing
$$\eta(z) = -(z-\alpha_1)\,(z-\alpha_2)\,(z-\alpha_3)\,(z-\alpha_4), \tag{52}$$

the convenient solution of (45) is

$$\theta(t) = \sum_{r=1}^{4}\{1/\eta'(\alpha_r)\}\,\mathrm{e}^{-\alpha_r t}\int_0^t \mathrm{e}^{\alpha_r x}P(x)\,dx. \tag{53}$$

Consequently
$$\theta(t+\tau) = \sum_{r=1}^{4}\{1/\eta'(\alpha_r)\}\,\mathrm{e}^{-\alpha_r(t+\tau)}\int_0^{t+\tau}\mathrm{e}^{\alpha_r y}P(y)\,dy. \tag{54}$$

In calculating $\lim\limits_{t\to\infty}\overline{\theta(t)\,\theta(t+\tau)}$ it is sufficient to consider $\lim\limits_{t\to\infty}\overline{\theta(t)\,\theta(t+|\tau|)}$ because the equality of these follows from the obvious relation

$$\lim_{t\to\infty}\overline{\theta(t)\,\theta(t+\tau)} = \lim_{t\to\infty}\overline{\theta(t)\,\theta(t-\tau)}. \tag{55}$$

The following relation is now required:

$$\lim_{t\to\infty}\mathfrak{W}\left[\mathrm{e}^{-\alpha_r t}\int_0^t \mathrm{e}^{\alpha_r x}P(x)\,dx\,.\,\mathrm{e}^{-\alpha_s(t+|\tau|)}\int_0^{t+|\tau|}\mathrm{e}^{\alpha_s y}P(y)\,dy\right] = \mathrm{e}^{-\alpha_s|\tau|}\frac{\overline{PP}}{\alpha_r+\alpha_s}. \tag{56}$$

To obtain this result one proceeds as in the derivation of (16). After transforming to a double integral in x and τ it will be found that the extra bit, t to $t+|\tau|$, in the range of the second integral on the left-hand side of (56) merely increases the range of integration in τ. As the integrand is essentially a δ-function and the range already includes the origin, $\tau = 0$, this contributes nothing to the integral. Hence the result (56).

One now obtains from (53), (54), (55) and (56)

$$\lim_{t\to\infty} \overline{\theta(t)\,\theta(t+\tau)} = \overline{PP} \sum_{r=1}^{4} \sum_{s=1}^{4} \frac{1}{\eta'(\alpha_r)\,\eta'(\alpha_s)} \frac{e^{-\alpha_r|\tau|}}{\alpha_r+\alpha_s}. \tag{57}$$

If only $\overline{\theta^2}$ is required, τ may be made zero in the above expression. The final result is obtainable on assuming that no factors consisting of the difference of roots can appear in the denominator of the final expression; this is plausible since there are physical situations corresponding to any pair of roots being equal and the result must be finite in such cases. The denominator is then of the form $\prod_{r\leqslant s} (\alpha_r+\alpha_s)$ and is of the tenth degree in the α's, so, considering the degree of (57), the numerator must be of the third degree. Since it is symmetric in the α's the general form can be written down in terms of β_1, β_2, ω_1 and ω_2. The unknown constants are determined by substituting various sets of numerical values for the α's. This gives (68).

Returning to the calculation of the correlation function, it is clear that the sum of those terms in the right-hand side of (57) for which r is 3 or 4 will, when expressed in terms of β_1, β_2, ω_1 and ω_2, differ from the sum of the remaining terms expressed in the same way, only in that the suffixes 1 and 2 will be interchanged. It will suffice, then, to fix attention for the moment on those terms for which r is 1 or 2.

Put

$$\left.\begin{aligned} \eta'(\alpha_1) &= A_1 - iB_1, \quad \eta'(\alpha_2) = A_1 + iB_1,\\ \eta'(\alpha_3) &= A_2 - iB_2, \quad \eta'(\alpha_4) = A_2 + iB_2, \end{aligned}\right\} \tag{58}$$

so that

$$\left.\begin{aligned} A_1 &= 4\omega_1'^2(\beta_1-\beta_2), \quad B_1 = 2\omega_1'\{(\beta_1-\beta_2)^2-(\omega_1'^2-\omega_2'^2)\},\\ A_2 &= 4\omega_2'^2(\beta_2-\beta_1), \quad B_2 = 2\omega_2'\{(\beta_2-\beta_1)^2-(\omega_2'^2-\omega_1'^2)\}. \end{aligned}\right\} \tag{59}$$

It is now easy to show that

$$\begin{aligned} \overline{PP} \sum_{r=1}^{2} \sum_{s=1}^{4} \frac{1}{\eta'(\alpha_r)\,\eta'(\alpha_s)} \frac{e^{-\alpha_r|\tau|}}{\alpha_r+\alpha_s} = \frac{2\overline{PP}}{A_1^2+B_1^2} e^{-\beta_1|\tau|} \Bigg\{ \sum_{s=1}^{4} \frac{\beta_1 A_1+\omega_1' B_1+\alpha_s A_1}{\eta'(\alpha_s)\,(\alpha_1+\alpha_s)\,(\alpha_2+\alpha_s)} \cos\omega_1'\,|\tau|\\ + \sum_{s=1}^{4} \frac{\beta_1 B_1-\omega_1' A_1+\alpha_s B_1}{\eta'(\alpha_s)\,(\alpha_1+\alpha_s)\,(\alpha_2+\alpha_s)} \sin\omega_1'\,|\tau| \Bigg\}. \end{aligned} \tag{60}$$

The sums appearing in (60) are obtained from the results

$$\sum_{s=1}^{4} \frac{1}{\eta'(\alpha_s)\,(\alpha_1+\alpha_s)\,(\alpha_2+\alpha_s)} = \frac{(\omega_2^2-\omega_1^2)+4\beta_1(\beta_1+\beta_2)}{4\beta_1\omega_1^2\{(\omega_2^2-\omega_1^2)^2+4(\beta_2+\beta_1)\,(\beta_2\omega_1^2+\beta_1\omega_2^2)\}}, \tag{61}$$

$$\sum_{s=1}^{4} \frac{\alpha_s}{\eta'(\alpha_s)\,(\alpha_1+\alpha_s)\,(\alpha_2+\alpha_s)} = \frac{-(\beta_1+\beta_2)}{2\beta_1\{(\omega_2^2-\omega_1^2)^2+4(\beta_2+\beta_1)\,(\beta_2\omega_1^2+\beta_1\omega_2^2)\}}. \tag{62}$$

These results are obtained by summing first the terms for which s is 1 or 2 and then those for which s is 3 or 4. Each pair is symmetric both in α_1 and α_2 and in α_3 and α_4.

Write

$$\left.\begin{aligned} Y_{12} &= (\omega_2^2-\omega_1^2)^2+8\beta_1^2\omega_2^2-4(3\beta_1^2-\beta_2^2)\,\omega_1^2+16\beta_1^2(\beta_1^2-\beta_2^2),\\ Z_{12} &= (\omega_2^2-\omega_1^2)\,(\omega_2^2-5\omega_1^2)+8\beta_1^2\omega_2^2-4(5\beta_1^2-3\beta_2^2)\,\omega_1^2+16\beta_1^2(\beta_1^2-\beta_2^2),\\ W &= (\omega_2^2-\omega_1^2)^2+4(\beta_2-\beta_1)\,(\beta_2\omega_1^2-\beta_1\omega_2^2),\\ X &= (\omega_2^2-\omega_1^2)^2+4(\beta_2+\beta_1)\,(\beta_2\omega_1^2+\beta_1\omega_2^2),\\ V &= \beta_2\omega_2^2+\beta_1\omega_1^2+4\beta_1\beta_2(\beta_1+\beta_2). \end{aligned}\right\} \tag{63}$$

Expressions denoted by Y_{21} and Z_{21} are obtained by interchanging the suffixes 1 and 2 in the expressions for Y_{12} and Z_{12} respectively. The values of the sums in (60) can now be written

$$\sum_{s=1}^{4} \frac{\beta_1 A_1 + \omega_1' B_1 + \alpha_s A_1}{\eta'(\alpha_s)(\alpha_1+\alpha_s)(\alpha_2+\alpha_s)} = \frac{\omega_1'^2}{2\beta_1\omega_1^2}\frac{Y_{12}}{X}, \tag{64}$$

$$\sum_{s=1}^{4} \frac{\beta_1 B_1 - \omega_1' A_1 + \alpha_s B_1}{\eta'(\alpha_s)(\alpha_1+\alpha_s)(\alpha_2+\alpha_s)} = \frac{\omega_1'}{2\omega_1^2}\frac{Z_{12}}{X}. \tag{65}$$

It is easy to show that

$$\left.\begin{aligned} A_1^2+B_1^2 &= 4\omega_1'^2 W, \\ A_2^2+B_2^2 &= 4\omega_2'^2 W. \end{aligned}\right\} \tag{66}$$

Combining the relation (60) with the corresponding expression for the terms in which r is 3 or 4, and substituting, one obtains the correlation function:

$$\lim_{t\to\infty}\overline{\theta(t)\,\theta(t+\tau)} = \frac{\overline{PP}}{4WX}\left[\frac{e^{-\beta_1|\tau|}}{\omega_1^2}\left\{\frac{Y_{12}}{\beta_1}\cos\omega_1'|\tau| + \frac{Z_{12}}{\omega_1'}\sin\omega_1'|\tau|\right\} + \frac{e^{-\beta_2|\tau|}}{\omega_2^2}\left\{\frac{Y_{21}}{\beta_2}\cos\omega_2'|\tau| + \frac{Z_{21}}{\omega_2'}\sin\omega_2'|\tau|\right\}\right]. \tag{67}$$

Putting $\tau = 0$ in this gives

$$\lim_{t\to\infty}\overline{\theta^2} = \frac{\overline{PP}}{4\beta_1\beta_2\omega_1^2\omega_2^2}\frac{V}{X}. \tag{68}$$

One now obtains for the autocorrelation function,

$$\lim_{t\to\infty}\frac{\overline{\theta(t)\,\theta(t+\tau)}}{\overline{\theta^2}} = \frac{\beta_1\beta_2}{VW}\left[e^{-\beta_1|\tau|}\omega_2^2\left\{\frac{Y_{12}}{\beta_1}\cos\omega_1'|\tau| + \frac{Z_{12}}{\omega_1'}\sin\omega_1'|\tau|\right\} + e^{-\beta_2|\tau|}\omega_1^2\left\{\frac{Y_{21}}{\beta_2}\cos\omega_2'|\tau| + \frac{Z_{21}}{\omega_2'}\sin\omega_2'|\tau|\right\}\right]. \tag{69}$$

Using (47) and (48), (68) becomes

$$\lim_{t\to\infty}\overline{\theta^2} = a^2\frac{kT\beta_1}{I_1\beta_1\beta_2\omega_1^2\omega_2^2}\frac{V}{X}. \tag{70}$$

Following convention we define $\overline{\theta^2}$ indirectly by stating the direct current which, when passed through the primary, would result in a steady deflexion in the secondary equal to the r.m.s. deflexion produced in it by Brownian fluctuations in the primary. This 'equivalent' current will be denoted by i_q. From (43) and (44), after replacing the right-hand side of (43) by $(G_1/I_1)\,i$, the relation between deflexion and steady current is

$$\theta = \frac{aG_1}{I_1\omega_1^2\omega_2^2}i, \tag{71}$$

so that

$$\lim_{t\to\infty}\overline{\theta^2} = \frac{a^2G_1^2}{I_1^2\omega_1^4\omega_2^4}i_q^2. \tag{72}$$

It follows from (72) and (70) that

$$i_q^2 = kT\frac{I_1\beta_1}{G_1^2}\frac{\omega_1^2\omega_2^2}{\beta_1\beta_2}\frac{V}{X}. \tag{73}$$

If the mechanical damping can be neglected $I_1\beta_1/G_1^2 \simeq 1/2R_1$. Otherwise it may be shown that

$$I_1\beta_1/G_1^2 = 1/\{2R_1(1-\epsilon)\}, \tag{74}$$

where ϵ is the ratio of mechanical damping to total damping, that is,

$$\epsilon = \kappa_1/(\kappa_1 + G_1^2/R_1). \tag{75}$$

Equation (73) now becomes

$$i_q^2 = \frac{kT}{2R_1(1-\epsilon)} \frac{\omega_1^2\omega_2^2}{\beta_1\beta_2} \frac{\beta_2\omega_2^2+\beta_1\omega_1^2+4\beta_1\beta_2(\beta_1+\beta_2)}{(\omega_2^2-\omega_1^2)^2+4(\beta_1+\beta_2)(\beta_1\omega_2^2+\beta_2\omega_1^2)}. \tag{76}$$

This has been written out in full because it is the result used in the experimental investigation. The units used will be amperes for i_q, ohms for R_1, joules for kT and radian/sec. for ω_1, ω_2, β_1 and β_2.

If k is assumed to be known*, all the quantities appearing in the right-hand side of (76) can be measured experimentally; the resulting value of i_q, together with a determination of the current sensitivity of the amplifier, gives the values of $\overline{\theta^2}$ to be expected on the assumption that the fluctuations are entirely thermal in origin.

Impulse calibration of amplifier

Using the result (76) to determine the expected value of $\overline{\theta^2}$ entails detailed knowledge of the characteristics of the two galvanometers; the calibration of the amplifier as a whole by determining its current sensitivity is merely one step among many. It may be said that this is because the response of the instrument to a steady current is not closely related to its response to the random forces appearing in the primary galvanometer. These random forces are more closely related to an impulse; both have a white spectrum. This suggests that the appropriate way to calibrate the amplifier, if one is bent on determining its thermal fluctuations, may be to measure its response to an impulse. It will be shown that, if this response is obtained, little further information is required to enable one to predict the value of $\overline{\theta^2}$ appropriate to thermal fluctuations, and that, moreover, the autocorrelation function for the fluctuations can be calculated from the record of the response alone.

Equations (43), (44) and (49) show that if a current $i(t)$ passes through the primary the deflexion in the secondary satisfies the equation

$$(D+\alpha_1)(D+\alpha_2)(D+\alpha_3)(D+\alpha_4)\,\theta = a(G_1/I_1)\,i(t). \tag{77}$$

If both galvanometers are initially at rest and a charge Q is passed impulsively through the primary, then, assuming that the random contribution to $i(t)$ is negligible, (77) shows that the subsequent motion of the secondary is given by

$$\theta(t) = aQ\frac{G_1}{I_1}\sum_{r=1}^{4}\frac{e^{-\alpha_r t}}{\eta'(\alpha_r)}. \tag{78}$$

It follows that
$$\int_0^\infty \theta(t)\,\theta(t+|\tau|)\,dt = a^2Q^2\frac{G_1^2}{I_1^2}\sum_{r=1}^{4}\sum_{s=1}^{4}\frac{e^{-\alpha_r|\tau|}}{\eta'(\alpha_r)\,\eta'(\alpha_s)\,(\alpha_r+\alpha_s)}. \tag{79}$$

Using (47) and (48) in (57) one obtains the following expression for the correlation function of the motion in the secondary arising from Brownian fluctuations in the primary:

$$\overline{\theta(t)\,\theta(t+\tau)} = 4kTa^2\frac{\beta_1}{I_1}\sum_{r=1}^{4}\sum_{s=1}^{4}\frac{e^{-\alpha_r|\tau|}}{\eta'(\alpha_r)\,\eta'(\alpha_s)\,(\alpha_r+\alpha_s)}. \tag{80}$$

Equations (79) and (80) give

$$\overline{\theta(t)\,\theta(t+\tau)} = 4kT\frac{I_1\beta_1}{G_1^2Q^2}\int_0^\infty \theta(t)\,\theta(t+|\tau|)\,dt, \tag{81}$$

* Alternatively, if the theory given above is assumed to be correct, the experiment could be used to determine k.

which by (74) becomes

$$\overline{\theta(t)\,\theta(t+\tau)} = \frac{2kT}{R_1(1-\epsilon)\,Q^2}\int_0^\infty \theta(t)\,\theta(t+|\tau|)\,dt. \tag{82}$$

Putting $\tau = 0$

$$\overline{\theta^2} = \frac{2kT}{R_1(1-\epsilon)\,Q^2}\int_0^\infty \theta^2(t)\,dt. \tag{83}$$

Van der Pol (1937) gave a result closely related to (83) in describing a method for calculating the fluctuations in voltage across two points in a network.

Now (82) and (83) give the autocorrelation function

$$\frac{\overline{\theta(t)\,\theta(t+\tau)}}{\overline{\theta^2}} = \frac{\int_0^\infty \theta(t)\,\theta(t+|\tau|)\,dt}{\int_0^\infty \theta^2(t)\,dt}. \tag{84}$$

Thus if one passes a known charge rapidly (i.e. in a time short compared with the response time) through the primary galvanometer and records the resulting throw in the amplifier, the autocorrelation function for the Brownian motion can be obtained from (84). It is then necessary to determine only the resistance in the input circuit and a small correction factor characteristic of the primary galvanometer to be able to use (83) to estimate the expected root-mean-square fluctuation of the amplifier. The ratio ϵ will normally be much less than unity, so its value need be known relatively imprecisely. In practice a device should be included in the amplifier to reduce a by a known factor without affecting the other parameters, so that the response can be determined for an impulse well above the level of Brownian fluctuations. This method of calibration was not used in the experimental investigation described in this paper, as it did not suggest itself till after the experimental work had been done. It is proposed to test the method experimentally later.

It is clear from their mode of deviation that the relations (83) and (84) will, with suitable modifications where necessary in the case of (83), apply very generally to linear instruments. The result (83) is reminiscent of Campbell's theorem, and, indeed, could be derived from this theorem very simply if the Brownian motion were regarded as having its origin in a random series of impulses. From this point of view (84) suggests a simple extension of Campbell's theorem which, we find, has been given by Rice (1944, 1945) and by Campbell & Francis (1946).

Part II. Experiment

The first object was to operate a galvanometer reliably so that over a long time interval (e.g. 30 min.) all other disturbances were small compared with thermal fluctuations. To achieve the required sensitivity, the deflexions of the primary galvanometer were amplified by introducing a split photocell system in the reflected primary beam, and measuring the current from this system by a secondary galvanometer.

Iron-selenium photocells were used without electronic amplification since this arrangement gave the best stability, but it necessitated using a sensitive secondary galvanometer whose period was comparable with that of the primary. This fact entailed the development of equation (76).

It was possible to test the detailed theory by varying such factors as the degrees of damping of primary and secondary galvanometers, and to confirm that Ising's, and not Astbury's, limit is the correct one.

Apparatus and method

The optical-lever amplifier developed for this investigation is described elsewhere (Jones 1951). It was designed to minimize thermal, mechanical, and electrical drifts, and such troubles as arise from convection currents in the optical path. Although all the quantitative results given in this paper apply to a particular galvanometer (a Tinsley type 4500) as primary, three such instruments were tried, and two of them reached the thermal limit with little difficulty. The galvanometers were tested both in air and *in vacuo*, and the Ising limit was attained in each case. Apart from showing directly, if demonstration were needed, that the presence of a gas around the suspended system does not affect the Brownian limit, this comparison also showed that, with the particular galvanometers used, there was negligible disturbance of the system by convection currents inside the housing. This confirmed an observation of Tear (1925) that the successful operation of light-pressure radiometers at atmospheric pressure depends much more on the design of the housing than on that of the suspended system; a small container, shrouded by metal walls, discourages the incidence of convection currents.

The amplifier and its associated equipment were operated in a temperature-stabilized room in the laboratory basement, where it was mounted on a concrete pillar set in the floor. The main workshop was only 30 ft. away, but when this was in operation it increased the recorded fluctuations by less than 10 % when the primary galvanometer was critically damped.

Rotation of the galvanometer mirror transferred light from one member to the other of a pair of iron-selenium photocells connected in parallel and in opposition. The difference current was recorded by a Kipp double-coil galvanometer (type Ka) and a Kipp recording camera. Typical extracts of records are shown in figure 1. The experimental procedure, after drifts had been substantially eliminated from the primary circuit and from the amplifier, was (i) to supply a known small ($\sim 10^{-8}$ V) calibration voltage to the primary circuit, (ii) to estimate the r.m.s. value of the fluctuations from the records, and (iii) to measure the constants of the two galvanometers and their associated circuits in order that these might be substituted in the theoretical formula (76) for the magnitude of the fluctuation, and the calculated value compared with the direct measurement. The overall accuracy aimed at was about 1 %, implying at least 5000 independent observations of the galvanometer trace (actually, about 7000 were made), and an accuracy of about 0·1 % in many of the steps in the calibration chain linking the observations with the laboratory standards of voltage, resistance and time.

Sensitivity calibration

The calibration voltage was developed from dry Leclanché cells through a chain of two potential dividers (attenuation of first step = 28·8, of second step = $4{\cdot}81 \times 10^6$) and was injected into the primary circuit across a resistance of 2·13 ohms in series with the galvanometer coil (10·76 ohms) and the main primary resistance (132·7 ohms). All these resistances were of copper. The voltage of the Leclanché cells was measured by a meter standardized

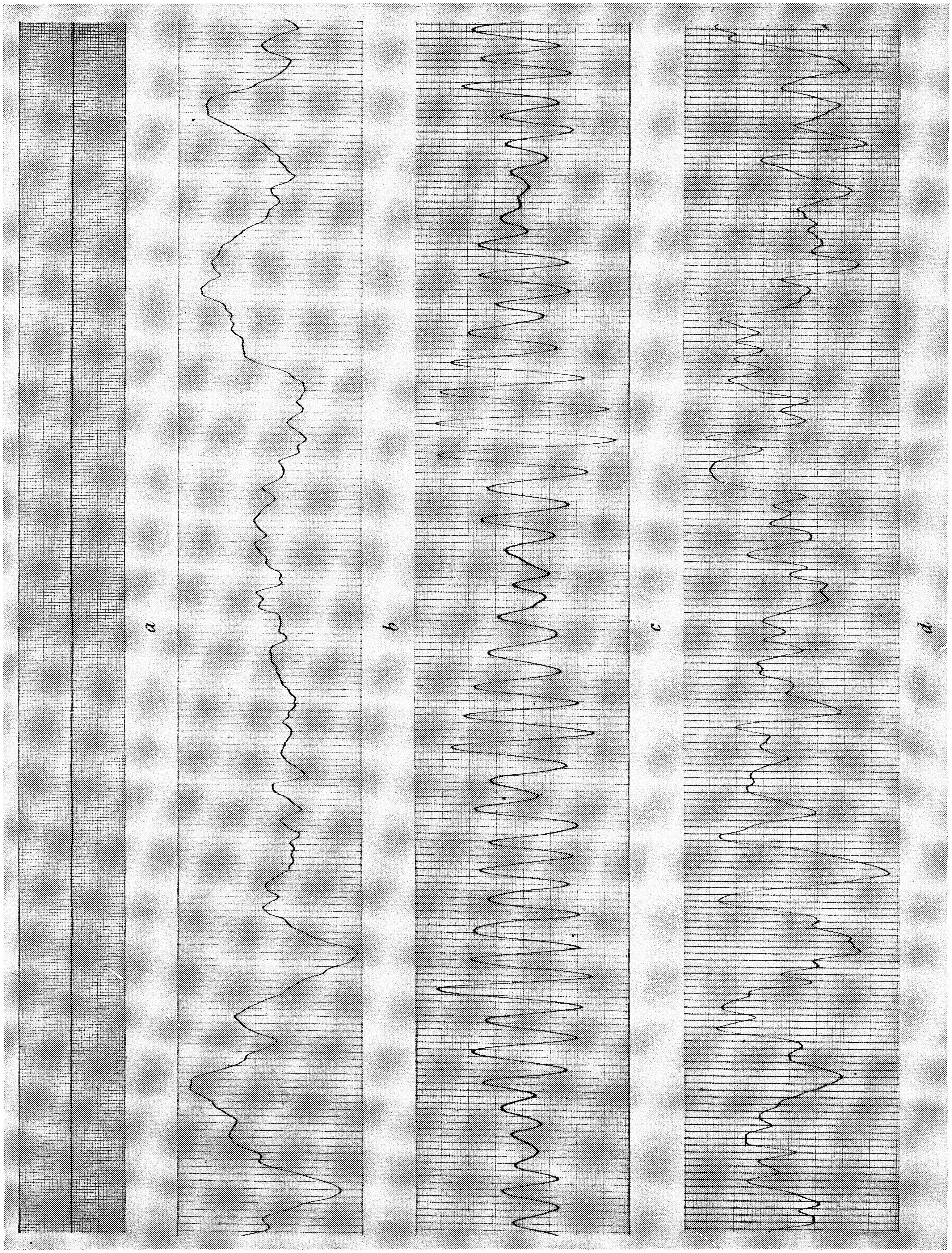

FIGURE 1. *a*. Record of secondary trace, with fixed primary mirror, at approximately twice the sensitivity used in records (*b*), (*c*) and (*d*). Vertical lines at 1 sec. intervals. *b*, *c*, *d*. Records of secondary trace with primary galvanometer (resistance 10·76Ω, free period 2·17 sec.) short-circuited (*b*), on open circuit (*c*), and nearly critically damped with external resistance 133·24 ohms (*d*). Secondary galvanometer (free period 1·60 sec.) nearly critically damped in each case. Sensitivity about $3 \cdot 2 \times 10^{-12}$ amp. per large division. Vertical lines at 0·5 sec. intervals.

against a Weston laboratory voltmeter; the resistances in the attenuator and in the primary circuit were measured on a Tinsley precision bridge, and corrected for a small change in temperature between standardizing and operating rooms.

It was important, with an attenuation of the order of 10^8, to check that there was no direct leakage from initial to final stages. Such a leakage did occur with the first attenuator constructed, and was discovered by the fact that the deflexions of the amplifier were not exactly proportional to the attenuation when the latter was varied in the ratios 1:2:3 by altering one resistance in the attenuator chain, although deflexions did vary correctly when one, two, and three cells respectively were connected in at the beginning of the chain. The leakage was eliminated by consistently earthing one side of each attenuator step, and also the metal box in which the attenuator was housed.

The voltage sensitivity of the system had to be determined when the recorded r.m.s. of the thermal fluctuations was about 15 mm., and the width of the recording paper only 120 mm. The procedure selected for speed, convenience, and appropriate accuracy was to attenuate the calibrating voltage to give a deflexion of about 800 mm. at the recording distance of about 1500 mm., and to determine this deflexion visually through the superimposed thermal fluctuations. The steps were as follows. A 1 m. scale was set up as near to the recording drum as possible. The zero of the secondary galvanometer was adjusted as closely as possible to the mid-point of the scale, and was then established more accurately through the amplified fluctuations of the primary system by finding, by trial and error, that scale reading for which the spot spent as much time on one side as on the other, as measured by stop-watch over the course of 1 min. The calibrating voltage was then applied to the primary circuit, and the new equilibrium position found by the same method. The calibrating cells were then reversed, and the third position found. The equality of the deflexions in either direction from zero was an incidental check on attenuator leakage. The observations were repeated several times, before and after the photographic recordings of amplified fluctuations. The results were consistent to within about $\pm 0{\cdot}3\,\%$.

Attention was paid to the following sources of error in the calibration procedure and appropriate corrections were applied where necessary:

(1) Non-linearity in the deflexions of the primary and secondary galvanometers, and in the optical lever amplifier, due to geometrical and other causes.

(2) Setting the recording drum and 1 m. calibrating scale parallel to one another and perpendicular to the line joining their mid-points to the secondary galvanometer mirror.

(3) The difference between the distances of calibrating scale and recording drum from the galvanometer mirror.

(4) A possible difference (none was found) between the divisions on the calibrating scale and those to be recorded on the photographic trace. Paper expansion in photographic processing, amounting to about 2 %, was immaterial since it affected equally the galvanometer deflexion and the recorded scale divisions.

It was imperative that the amplifier performance should remain sensibly unchanged between calibrations. The amplification might have altered slightly owing to many causes, the most serious being variation in lamp brightness, since the amplification was roughly proportional to the 2·5 power of the lamp voltage. Other causes, such as variation in photocell behaviour with temperature, were minimized by stabilizing the temperature to better than 0·2° C. The lamp voltage was kept constant by large-capacity batteries to better than 0·1 % in any one run of an hour and to 0·3 % between any two runs.

The overall error in measuring and preserving the sensitivity of the complete system, when operating at about 3×10^{-13} amp./mm. and when the foregoing procedure and precautions had been adopted, was estimated to be compounded from about ±0·75 % standard error in the calibrating voltage and about ±0·5 % standard error in measuring the deflexion of the system, giving about ±0·9 % standard error overall. In some of the earlier measurements (those subsequently described as cases *A*, *B* and *C*), a further error might have occurred, since a small contact resistance was subsequently found in the primary circuit; the uncertainty in this resistance at the time of the experiments was allowed for by increasing the estimate of the overall standard error for cases *A*, *B* and *C* to ±1·2 %.

Statistical analysis of records

The r.m.s. deflexion of the secondary galvanometer was to be statistically estimated from data obtained from photographic records of the type shown in figure 1. The amplification was adjusted to produce an r.m.s. deflexion in the secondary galvanometer of about 15 mm., since this was the largest value for which the trace would stay within the 120 mm. width of the record. The records were marked by longitudinal lines at 1 mm. intervals, to enable the deflexion at any instant to be measured, and by transverse lines at 0·5 sec. intervals. The recording paper speed was about 1 cm./sec., giving a record on which the intersection of the galvanometer trace with any time line could be estimated if necessary to 0·1 mm. The deflexions of the secondary galvanometer were then read from some arbitrary fixed zero on the record, usually at 1·5 sec. intervals. The limit to the number of successive observations (900 to 1000) was set up by the paper capacity of the recorder, which corresponded to a run of about 25 min. The reading interval was chosen as being the minimum one for which correlation between successive readings was not unduly high.

The estimation of the r.m.s. deflexion from the records was complicated by the coexistence of the foregoing correlation with an inevitable small drift due to thermal and other causes in the primary circuit. It was therefore necessary to estimate the drift, and hence the largest period over which it could be safely neglected in the ensuing statistical analysis.

The data from any one record were divided into groups, each of N successive deflexions, and were analyzed in the following manner. Suppose that in any group the rth deflexion from the arbitrary zero (e.g. the zero of the deflexion scale printed upon the recording paper) is of magnitude x_r, and that the deflexion from the true zero (i.e. the position of the secondary galvanometer corresponding to zero current in the primary) at the same instant is X_r. It is $\overline{X_r^2}$, or $\overline{X^2}$, that has to be determined. The autocorrelation coefficients are of the form

$$\rho_s = \frac{\overline{X_r X_{r+s}}}{\overline{X^2}},$$

and would all be zero, of course, if there were no correlation between successive observations. On the supposition that the drift stays constant inside any one group of N observations, the true zero at the rth instant is given by $(a+rd)$, where a and d are constants for that group.

If this be so, and $\mathfrak{W}$ denotes an average taken over the groups of N, then the basic statistical result employed is

$$\frac{1}{N-1}\mathfrak{W}\left\{\sum_1^N x_r^2-\frac{1}{N}\left(\sum_1^N x_r\right)^2\right\}=\overline{X^2}\left\{1-\frac{2}{N(N-1)}\sum_{s=1}^{N-1}(N-s)\,\rho_s\right\}+\tfrac{1}{12}N(N+1)\,\mathfrak{W}d^2. \quad (85)$$

The terms on the right-hand side of (85) show the relative effects on the value of $\overline{X^2}$ of the autocorrelation coefficients ρ_s and of the drift factor d. The former can be estimated by substituting in equation (69) the known values of the recording time interval τ and the galvanometer conditions; usually only ρ_1 and ρ_2 are appreciable. The values obtained, however, depend rather critically on the damping of the galvanometers, and N should therefore be chosen large enough for the contribution of the correlation terms to be small. Too large a value of N must not, however, be chosen, otherwise the drift contribution may become appreciable, and $\mathfrak{W}d^2$ cannot be determined accurately without recourse to advanced statistical method. It can be estimated roughly by examining the averages of the consecutive groups of N readings, and looking for variations in these averages which show through the expected random fluctuations $\sqrt{(\overline{X^2})}/\sqrt{N}$.

The deviations (in mm.) of the means of successive hundreds of readings from the mean of all the readings on the record are given in table 1 for two records, E_1 and A_3, selected from the records used as showing the greatest and least drifts respectively. A_3 is more typical than E_1. The r.m.s. Brownian movement on both records was about 15 mm.

Table 1

E_1	−13·3	−16·9	−6·1	+1·5	+7·2	+7·7	+7·9	+5·9	
A_3	− 1·9	− 0·1	+3·0	−1·5	+3·3	+1·1	+2·3	−1·9	+0·7

Record A_3, lasting 25 min., shows the extent to which drifts could be reduced in favourable circumstances. Over considerable periods on several records there was no drift significant above the Brownian fluctuations.

The drift during the time of 100 observations seldom approached 10 mm., and it may therefore be reasonably assumed that the value of d to be used in (85) is always less than 0·1. If this is so, and N is made equal to 20, the drift contribution will increase the mean-square deflexion of about 15^2 mm.^{2} by less than 0·3 mm.^{2}, giving an increase of less than 0·1 % in the r.m.s. value. Consequently, if N is made equal to 20, drift may be ignored.

With groups of 20, the correlation term in (85) is about 2 or 3 %, at least when neither galvanometer is far from critical damping. The value determined from the experimental values of the galvanometer parameters in (69) should therefore be sufficiently accurate. In cases where the primary galvanometer was seriously under- or over-damped, the correlation was kept sufficiently small by increasing the reading time interval. The drift was still small enough to permit this to be done.

A more thorough statistical analysis of the records was made by Mr M. H. Quenouille. In this the autocorrelation coefficients were assumed to be known only in so far as they might be determined by the records themselves, as was indeed the case when most of the records were first analyzed. The method used consisted essentially in estimating the means of the sums appearing on the left in (85) for three values of N (4, 20, 100) for each record and applying analysis of variance to the results. The estimates of $\overline{X^2}$ made in this way did not differ appreciably, in most cases, from the result obtained by the more straightforward method, although it will be seen that they tend to be a little lower (table 2).

In assessing the results, Mr Quenouille's values for $\overline{X^2}$ have been used; but sufficient records have been analyzed in the elementary way described to establish the substantial equivalence of the two methods.

The r.m.s. deflexions in mm. from each record were translated into terms of equivalent current. The resulting values are tabulated in table 2, but before examining them we shall consider the experimental determination of the physical quantities necessary to the evaluation of the fluctuations expected from theory.

Establishment of physical quantities required by the theoretical formula

The quantities to be evaluated were the resistances of primary and secondary circuits, and the periods and damping characteristics of each galvanometer, for the several conditions of experiment.

The resistances were all measured to 0·1 %, except for that of the photocell combination, which helped to damp the secondary galvanometer. This resistance had to be measured under the actual conditions of operation, with the light from the primary galvanometer falling on the cells, and with the secondary galvanometer connected across them. The measurement was easiest made by injecting a small e.m.f. (4×10^{-3} V) into the secondary circuit, reversing it, and measuring the deflexion of the galvanometer. A resistance was then inserted instead of the cells, and a value found that gave the same deflexion; measurements made by this method were consistent to better than 1 %. With one pair of cells it was found that the equivalent resistance was different when the primary mirror was fixed (8600 ohms) from its value when the mirror was executing Brownian fluctuations (8100 ohms). In the former case the pattern of light on the photocells was, of course, stationary; in the latter it was oscillating with about $0{\cdot}15\mu$ r.m.s. fluctuation. This curious effect was repeated at least three times with one pair of cells, but disappeared with cells of another make; it made the establishment of the cell resistances more tedious, since the deflexions of the secondary galvanometer had to be observed through the amplified fluctuations of the primary.

The determination of the galvanometer characteristics was carried out, as was the attenuator construction and calibration, by Mr J. C. S. Richards. His method was as follows:

It is sufficient to know the resistances, free periods, damping constants on open circuit, and critical damping resistances. Of these, all but the last may readily be determined with sufficient accuracy, but it is not easy to determine the critical damping resistance to within 1 % by the usual trial-and-error method. A better determination is possible from observations on the galvanometer in under-damped conditions.

If an under-damped galvanometer with no e.m.f. in its circuit is swinging, the approach of the deflexion θ to zero is given by an equation of the form

$$\theta = C\,e^{-\beta t}\cos(\omega' t-\gamma), \tag{86}$$

where C and γ are constants and the notation is otherwise that of equations (43), (44) and (50) with the suffixes dropped. Observation of the logarithmic decrement δ (i.e. the natural logarithm of the ratio of the amplitudes of successive swings to either side of zero) enables β/ω to be determined, since (86) gives

$$\beta/\omega = \delta/\sqrt{(\pi^2+\delta^2)}. \tag{87}$$

Now β is the sum of β_A and β_R, where β_A is the mechanical damping term $\kappa/2I$, which does not vary with R, and β_R is the electromagnetic damping $G^2/2RI$. If the decrement is observed with the galvanometer on open circuit, β_A/ω is obtained. Then observation of the decrement when the circuit resistance has a value R, greater than the critical resistance R_c, allows $(\beta_A+\beta_R)/\omega$ and hence β_R/ω to be determined. Since β_R is inversely proportional to R, the critical damping resistance can be determined from the condition that $(\beta_A+\beta_{R_c})/\omega$ is unity. Conversely, if β_A/ω and R_c are known the value of β/ω can be found for any value of R.

If the open-circuit period is now observed, ω can be determined from the relation

$$\omega'/\omega = \pi/\sqrt{(\pi^2+\delta^2)}. \tag{88}$$

As an example, the observations are given for the primary galvanometer used in the investigation, when evacuated:

galvanometer resistance = 10·76 Ω
free period = 2·17 sec.

external resistance	δ	β/β_c	R_c (calculated)
∞	0·0858	0·0273	—
4000	0·1970	0·0627	145·5
2000	0·3048	0·0967	143·2
1000	0·5330	0·1673	145·4
500	0·9944	0·3018	144·2
250	2·1305	0·5610	143·0

$$\omega = 2{\cdot}895\ \text{sec.}^{-1}$$

$$R_c = 144{\cdot}3\ \Omega$$

$$\text{actual value of } R = 149{\cdot}5\ \Omega$$

$$\text{actual value of } \beta = \beta_A + \frac{R_c}{R}(\omega-\beta_A)$$

$$= 2{\cdot}811\ \text{sec.}^{-1}$$

One obtains thus the values of β_1, β_2, ω_1, ω_2 and ϵ for substitution in (76). The resistance [redacted]

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In the second group, the secondary galvanometer had constant damping, which was very nearly critical, and nearly all the records were taken when the primary galvanometer too was very nearly critically damped (condition E). Two other records were taken, however, one with the primary galvanometer on open circuit (D), and one with it short-circuited (F). The relevant values for this group were:

primary circuit		secondary circuit
$\omega_1 = 2{\cdot}895\,\text{sec.}^{-1}$		$\omega_2 = 3{\cdot}927\,\text{sec.}^{-1}$
$R_{c,1} = 144{\cdot}3\,\Omega$		$R_{c,2} = 2882\,\Omega$
$R_1 = \infty,$	$\epsilon = 1{\cdot}0\,(D)$	$R_2 = 2880\,\Omega$
$= 144{\cdot}0\,\Omega,$	$\epsilon = 0{\cdot}0268\,(E)$	
$= 10{\cdot}76\,\Omega,$	$\epsilon = 0{\cdot}0021\,(F)$	

To obtain a theoretical value for the equivalent current (i.e. the steady current which would produce a deflexion equal to the r.m.s. Brownian motion deflexion), the above values were substituted in equation (76), for all conditions except the open-circuit case (D), where equation (73) had to be employed. The results are shown in table 2, where they are compared with those found by direct measurement from the experimental records. Even if the standard error for every one of the quantities listed above were 1 %, the standard error in the value of i_q calculated from them would be only about 0·7 %, at least when both galvanometers are nearly critically damped. The theoretical figures are therefore unlikely to possess a standard error of more than 0·5 %.

Detailed comparison between theory and experiment

The experimental results are set out and compared with the theoretical values in table 2. Column 1 gives the values determined using Mr Quenouille's estimates (with standard errors) of the r.m.s. deflexions on the records, while column 2 gives the values consequent on the more elementary analysis of the results. Column 3 gives the effective number of observations. Column 4 gives the theoretical values. Column 5 gives the values in column 1 as percentages of the theoretical values, while column 6 gives the weighted mean of the percentages (with its standard error) for each of the two sets of experiments.

Table 2

(Units 10^{-12} amp.)

run	1	2	3	4	5	6
A_1	5·22 ± 0·12	5·15	855	5·21	100·1 % ± 2·3 %	99·9 % ± 1·1 %
A_2	4·86 ± 0·14	5·04	600	5·21	93·3 % ± 2·7 %	
A_3	5·41 ± 0·12	5·53	950	5·21	103·7 % ± 2·3 %	
B_1	5·25 ± 0·16	—	525	5·52	95·1 % ± 2·9 %	
B_2	5·54 ± 0·18	—	450	5·52	100·4 % ± 3·3 %	
C_1	5·97 ± 0·17	—	600	5·78	103·2 % ± 2·9 %	
D_1	4·84 ± 0·24	—	200	4·25	(113·9 % ± 5·6 %)	
E_1	5·34 ± 0·13	5·38	760	5·45	96·9 % ± 2·4 %	100·1 % ± 1·3 %
E_2	5·62 ± 0·14	5·51	760	5·45	103·2 % ± 2·6 %	
E_3	5·48 ± 0·16	—	570	5·45	100·6 % ± 2·8 %	
E_4	5·43 ± 0·16	—	570	5·45	99·7 % ± 2·8 %	
F_1	6·05 ± 0·32	—	175	6·34	95·4 % ± 5·0 %	

For the runs analyzed in both ways Mr Quenouille's estimate of the r.m.s. deflexion is, on average, 0·5% lower than that given by the elementary analysis: the difference is not significant.

It will be seen that the Brownian motion varies in magnitude approximately in the way indicated by equation (76), although its value is rather high in the case of the extremely underdamped primary, in which case the instrument would, of course, be very sensitive to external disturbance. In the case of an impulsive disturbance this is clear from (81) with $\tau = 0$; for a given impulsive disturbance corresponds to passing a given charge through the coil and when Q is fixed the ratio of the integrated square of the throw to $\overline{\theta^2}$ is proportional to $1/\beta_1$. So a given impulsive disturbance would in the open circuit case be about 35 times more effective in increasing the apparent value of $\overline{\theta^2}$ than it would be in the critically damped case.

Since the experimental result is only 14 % too high in the former case, it may be concluded that if the disturbance causing this may be treated as a series of impulses acting independently it will increase the critically damped value by only about 0·3 %.

Discussion of results

In calculating the overall agreement between expected and observed values, the record D_1 has been excluded, since external vibration could have contributed appreciably to the observed fluctuation; it is perhaps surprising that the contribution was as small as 14 %, but the record was a selected one taken in a quiet period.

The stated standard errors only take account of errors arising from the statistical analysis of the records. Errors may also have arisen in measuring the necessary physical quantities from which the theoretical values were calculated, and in determining the sensitivity of the system. The theoretical figures are unlikely to possess a standard error of more than 0·5 %. The determination of the current sensitivity involved standard errors of 1·2 % in cases A, B, C and of 0·9 % in cases D, E and F. The cumulative effect of these errors is to make the mean of the ratios of experimental to theoretical figures 100·0 % with a standard error of ±1·1 %.

It can therefore be concluded that Ising's limit is the correct one, that the extended theory given in this paper is experimentally confirmed, that galvanometers can be reliably operated at the theoretical limit, and that drifts and fluctuations due to external causes can be made small compared with those arising from thermal agitation.

We thank Mr J. C. S. Richards and Mr M. H. Quenouille for their contributions, already mentioned in the appropriate parts of this paper. We also thank Mr D. C. Gall of Messrs H. Tinsley and Company, for his co-operation in providing galvanometer movements for the primary circuit, and Professor A. Michels for his help in obtaining a suitable galvanometer for the secondary circuit. We are indebted to Mr W. C. Jolly for checking our computations.

Appendix

Since the work described in this paper was completed some further investigations have been carried out on the records of amplifier fluctuations. Rice (1945) has shown that, in a one-dimensional Gaussian random process with autocorrelation coefficient $\rho(\tau)$, the number of times per second that the deflexion becomes zero, say N_1, is given by

$$N_1 = \frac{1}{\pi}[-\rho''(0)]^{\frac{1}{2}},$$

and the number of times per second that the rate of change of the deflexion with respect to time becomes zero, say N_2, is given by

$$N_2 = \frac{1}{\pi}\left[\frac{\rho^{\text{iv}}(0)}{-\rho''(0)}\right]^{\frac{1}{2}}.$$

In both cases dashes denote differentiation with respect to τ.

Some doubt has been cast on the general validity of these results: Uhlenbeck, quoted by Rice (1945), has pointed out that, for a suspended mirror, the above result makes N_1 independent of the damping, i.e. of the pressure of the surrounding gas, whereas Kappler's (1931) records of the motion of a suspended mirror vary markedly in character with the pressure. It is not, however, possible actually to assess N_1 from Kappler's published records. The fact that we have obtained very satisfactory verification of Rice's formulae in the closely related case of the galvanometer amplifier may, therefore, be of some interest.

Using the expression obtained in the above paper, equation (69), for the autocorrelation function of the fluctuations of the galvanometer amplifier, Rice's results become

$$N_1 = \frac{1}{\pi}\left\{\frac{\omega_1^2\omega_2^2(\beta_1+\beta_2)}{\beta_2\omega_2^2+\beta_1\omega_1^2+4\beta_1\beta_2(\beta_1+\beta_2)}\right\}^{\frac{1}{2}},$$

$$N_2 = \frac{1}{\pi}\left\{\frac{\beta_2\omega_1^2+\beta_1\omega_2^2}{\beta_1+\beta_2}\right\}^{\frac{1}{2}}.$$

These results were tested on records obtained with the galvanometer conditions denoted by D, E and F in the paper. In each of these the secondary galvanometer was critically damped, while the primary was open-circuited in case D, critically damped in case E, and short-circuited in case F. The actual position of the zero line on the trace was not known and instead the mean of a series of consecutive observations had to be used. If this series did not cover a sufficient length of the record the mean would not lie close to the true zero and, in general, too large a value would be obtained for N_1; if, on the other hand, the series were too long, drift in the true zero would result in too small a value being obtained. In case F the mean of 600 observations at 1·5 sec. intervals had to be taken as zero. When the mean of 80 observations was used the value obtained for N_1 was about 20 % too high: this was not surprising since it was clear from the records for this case (see figure 1*b*) that the nature of the fluctuations was such that the trace stayed well to one side of the zero line for considerable periods. However, measuring deflexions from the mean of 600 observations gave almost the theoretical value for the r.m.s. Brownian motion deflexion, so drift was presumably negligible in a run of this length. The other cases presented no difficulty.

In the estimation of N_2, case F was again the only one to give any trouble. The slope of the trace often stayed near zero for distances on the record corresponding to two or three seconds, and it was difficult to judge how often it had actually been zero.

	no. of zeros per sec. (N_1)		no. of points of zero slope per sec. (N_2)	
case	exp.	theor.	exp.	theor.
D	$0{\cdot}88 \pm 0{\cdot}04$	0·890	$0{\cdot}925 \pm 0{\cdot}03$	0·929
E	$0{\cdot}46 \pm 0{\cdot}015$	0·478	$1{\cdot}08 \pm 0{\cdot}03$	1·073
F	$0{\cdot}15 \pm 0{\cdot}015$	0·147	$1{\cdot}12 \pm 0{\cdot}04$	1·222

The errors indicated are the standard errors appropriate to Poisson fluctuations in the number of zeros, or points of zero slope, counted in each case. Since the actual distributions might not follow the Poisson law, the foregoing figures can be taken only as rough estimates of the true standard errors.

[*Note added to proof*, 21 *November* 1951.] The impulse method of calibration mentioned in the text has been tested experimentally, and has given values for i_q agreeing to within about 1 % with those obtained from (76).

Relatively coarse grids (see Jones 1951) were used in the amplifier to decrease the sensitivity and to increase the range of linear response. The impulsive current was obtained by discharging a mica condenser into the primary circuit, and the response of the system to the impulse was recorded photographically. Three cases were tested: (i) a single galvanometer, critically damped, (ii) an amplifier with both galvanometers critically damped, and (iii) an amplifier with the primary galvanometer on open-circuit and the secondary critically damped.

Values of i_q were obtained from the recorded responses using the result

$$i_q^2 = \frac{2kT}{R_1(1-\epsilon)} \frac{\int_0^\infty \theta^2(t)\,dt}{\left\{\int_0^\infty \theta(t)\,dt\right\}^2}$$

which follows from (83) since the time integral of the throw is just the charge passed multiplied by the current sensitivity.

The accuracy of the method depends largely on keeping the discharge time small and on evaluating the integrals precisely. By varying the time constant and the time for which the condenser was in circuit, the finite discharge times actually used (roughly 0·001 of the response time) were shown to affect the estimate of i_q by less than 0·5 %. In cases (i) and (ii), the error arising from numerical evaluation of the integrals did not exceed 0·5 % and in these cases it was unnecessary to know either the magnitude of the charge passed or the current sensitivity. In case (iii), however, where the record was long and oscillatory, an error of only 0·1 mm. in the position of the zero line, while affecting the integral of $\theta^2(t)$ by less than 0·1 %, would have changed the integral of $\theta(t)$ by as much as 12 %. Consequently, in this case both the charge passed and the current sensitivity were measured, and (83) used directly.

Values of the autocorrelation function obtained from the impulsive responses by using (84) agreed to within about 1 % with the values from (69).

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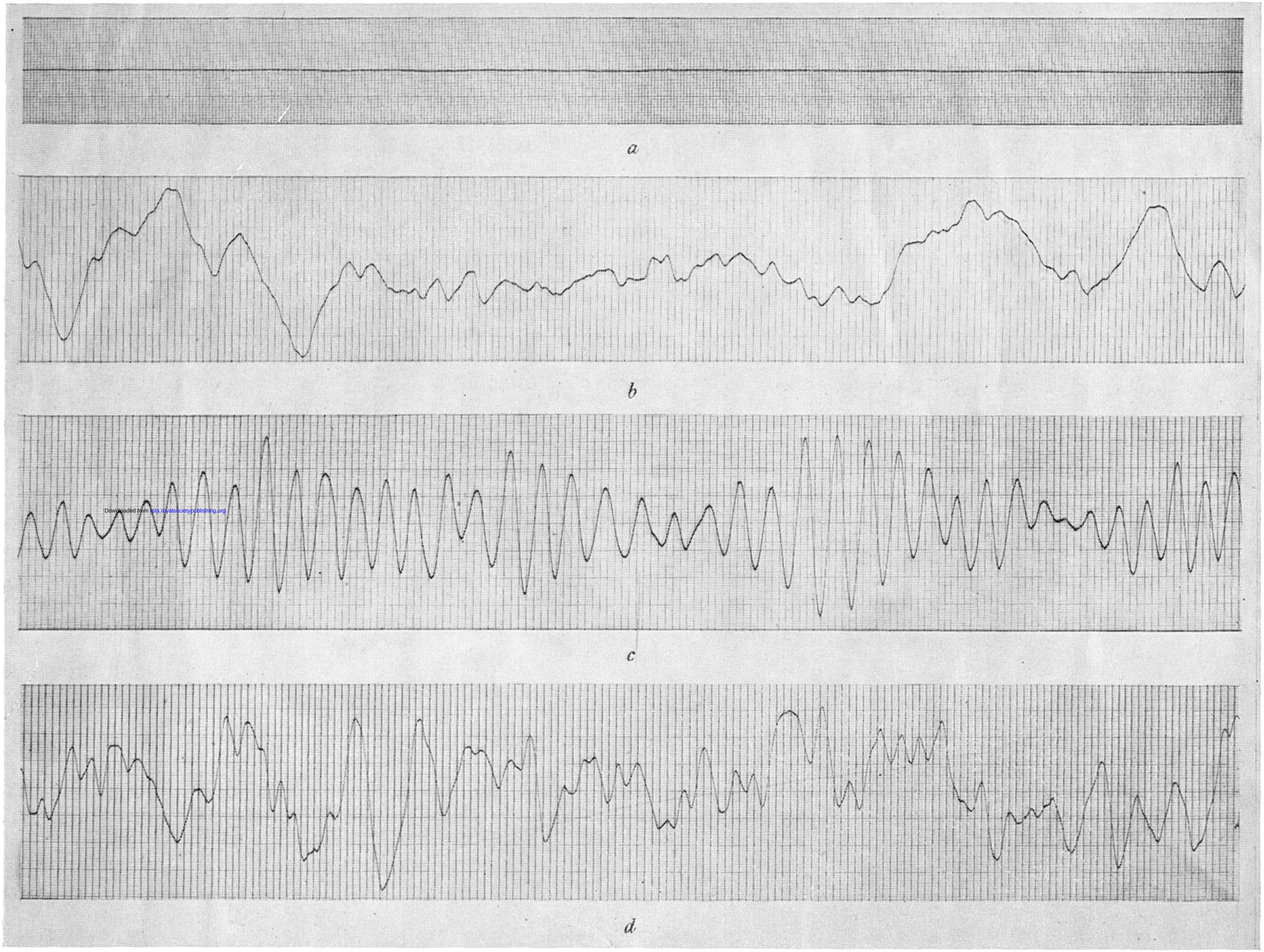

Figure 1. *a*. Record of secondary trace, with fixed primary mirror, at approximately twice the sensitivity used in records (*b*), (*c*) and (*d*). Vertical lines at 1 sec. intervals. *b*, *c*, *d*. Records of secondary trace with primary galvanometer (resistance 10·76Ω, free period 2·17 sec.) short-circuited (*b*), on open circuit (*c*), and nearly critically damped with external resistance 133·24 ohms (*d*). Secondary galvanometer (free period 1·60 sec.) nearly critically damped in each case. Sensitivity about $3{\cdot}2\times10^{-12}$ amp. per large division. Vertical lines at 0·5 sec. intervals.